Odd-Order Variable-Bandwidth Digital Filter Using Lp-Norm-Error Minimization

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***Abstract*—This paper details a 2-step tactic for implementing an odd-order (OO) variable-bandwidth (VBW) filter with absolutely guaranteed stability. The design scheme minimizes the Lp-norm of weighted amplitude-response-error vector, and the OO-VBW filter coefficients are formed using polynomials in the bandwidth (BW)-varying parameter. Since those coefficients have variable function values, the designed OO-VBW filter possesses variable amplitude response. Another crucial issue discussed in this paper is about stability guarantee. Since changing the filter coefficients (in terms of the polynomial-function values) may cause instability, the stability must be ensured during online tuning. A coefficient-conversion (CC) technique is employed for achieving this objective. A bandpass OO-VBW filter is simulated for validating the presented 2-step tactic as well the stability guarantee.**

***Keywords*—Digital filter, variable digital filter, odd-order (OO), variable-bandwidth (VBW), stability.**

I. Introduction

Variable-bandwidth (VBW) filters possess the capacity to vary frequency bandwidths such as passband width, stopband width, and transition bandwidth. Such VBW filters are needed in a wide range of fields, which include various signal processing fields and digital communications. For instance, such VBW filters are required in selecting frequency bands according to the practical needs. With such tunable bandwidths, one can flexibly change the bandwidths on demand. A recursive variable filter requires much less computational complexity and much less hardware-realization cost than its nonrecursive counterpart [1]-[10], but has the possibility to become unstable [11]-[17]. For this reason, the first priority in recursive VBW filter design is to guarantee its stability such that the online tuning process will not risk the instability. If the stability is not guaranteed, the VBW filter is not applicable to the real-time tuning. An unstable VBW filter is useless because it makes filtering outputs diverge.

In this paper, a coefficient-conversion (CC) technique is employed in the process of designing a stable odd-order (OO) VBW filter (OO-VBW filter). The CC technique carries out coefficient conversions on the coefficients of the OO-order recursive VBW filter in such a manner that the conversions theoretically ensure its stability. This paper first formulates designing an OO-VBW filter using a 2-step tactic, which incorporates the CC technique in the 2 design steps, and the least pth error criterion is used. The two basic steps involve designing odd-order constant-bandwidth (OO-CBW) recursive filters, and then fitting polynomials to the resulting coefficient values. In the 2 steps, the CC technique is incorporated, aiming to ensure the stability. That is, the first step with the CC technique incorporated yields stable OO-CBW filters, and the second step yields a stable OO-VBW filter. In both the 2 steps, the CC technique is crucial to guaranteeing the stability (OO-CBW filter's stability and OO-VBW filter's stability). This approach can be viewed as a generalized design from the one in [15], aiming to design a stable OO-VBW filter. A bandpass OO-VBW filter is designed and the design results are given for exemplifying the 2-step tactic and showing the ensured stability of the resultant OO-VBW recursive filter.

II. Stabilized Variable Model

An OO-VBW filter takes the form

$$H(z,\rho)=\frac{A(z,\rho)}{B_0(z,\rho)\prod_{i=1}^{N_2}B_i(z,\rho)} \tag{1}$$

with

$$\begin{aligned} A(z,\rho)&=\sum_{i=0}^{N_1}a_i(\rho)z^{-i}\\ B_0(z,\rho)&=1+b_{01}(\rho)z^{-1}\\ B_i(z,\rho)&=1+b_{i1}(\rho)z^{-1}+b_{i2}(\rho)z^{-2}. \end{aligned} \tag{2}$$

Here, $B_0(z,\rho)$ is the first-order section of the transfer functions's denominator, and $B_i(z,\rho), i=1,2,\cdots,N_2$, denote the 2nd-order blocks involved in $H(z,\rho)$'s denominator. Moreover, ρ represents a parameter for tuning $H(z,\rho)$'s bandwidths, $\rho\in[\rho_{\min},\rho_{\max}]$. Here, we notice that the coefficients

$$a_i(\rho), b_{01}(\rho), b_{i1}(\rho), b_{i2}(\rho)$$

are parameterized as the functions of ρ. Thus, the odd-order model (1) has variable coefficients. The above model can be modified to

$$H(z,\rho)=\frac{\sum_{i=0}^{N_1}a_i(\rho)z^{-i}}{\prod_{i=0}^{N_2}[1+b_{i1}(\rho)z^{-1}+b_{i2}(\rho)z^{-2}]} \tag{3}$$

with

$$b_{02}(\rho)=0.$$

To ensure the stability of $H(z,\rho)$, we must guarantee

$$\begin{cases}|b_{i2}(\rho)| < 1 \\ |b_{i1}(\rho)| < 1 + b_{i2}(\rho).\end{cases} \tag{4}$$

To enforce the condition in (4) to be always satisfied, we can convert the original $b_{i2}(\rho)$ (except for $b_{02}(\rho)=0$), $b_{i1}(\rho)$ to $x_{i2}(\rho), x_{i1}(\rho)$ through the coefficient-conversions

$$\begin{cases}b_{i2}(\rho) = \gamma \cdot C(x_{i2}(\rho)) \\ b_{i1}(\rho) = \gamma \cdot C(x_{i1}(\rho))[1 + b_{i2}(\rho)]\end{cases} \tag{5}$$

where the function $C(x)$ must satisfy

$$C(x) \in [-1, 1] \tag{6}$$

and the scaling factor γ must be

$$\gamma \in (0, 1) \tag{7}$$

Like $b_{i2}(\rho)$ and $b_{i1}(\rho)$, $x_{i2}(\rho), x_{i1}(\rho)$ are also the polynomials in ρ. We can prove that the required condition (4) for stability is always ensured if the function $C(x)$ satisfies the condition in (6) and γ satisfies the requirement in (7).

III. Odd-Order L_p-Norm Design

The target of an OO-VBW filter design is to determine a variable model (3) so as to best approximate the desired amplitude-response $A_d(\omega,\rho)$, where $0 \le \omega \le \pi$, and $\rho_{\min} \le \rho \le \rho_{\max}$. This goal is achieved by adopting a two-stage procedure employed in [15]. That is, the first stage is to discretize the whole range of ρ such that the discrete-points ρ_n, $n = 1, 2, \cdots, N$, are obtained. Then, the discretized $A_d(\omega, \rho_n)$, which is associated with ρ_n, is taken as the design specification of an OO-CBW filter

$$H(z) = \frac{\sum_{i=0}^{N_1} a_i z^{-i}}{\prod_{i=0}^{N_2} [1 + b_{i1} z^{-1} + b_{i2} z^{-2}]}. \tag{8}$$

Similarly, the original denominator-coefficients b_{i2}, b_{i1} must be converted to other parameters x_{i2}, x_{i1} for the stability reason. Specifically, b_{i2}, b_{i1} are expressed by using the conversion function $C(x)$ as

$$b_{i2} = \gamma \cdot C(x_{i2})$$
$$b_{i1} = \gamma \cdot C(x_{i1})(1 + b_{i2}).$$

Suppose that p is a predetermined number, for example, $p = 10$. The coefficients $\{a_i, x_{i2}, x_{i1}\}$ are determined by minimizing the L_p-norm

$$e_p = \|\boldsymbol{e}\|_p \tag{9}$$

with

$$\begin{aligned}\boldsymbol{e} &= \boldsymbol{W}(\boldsymbol{f} - \boldsymbol{g}) \\ \boldsymbol{f} &= \begin{bmatrix} f_1 & f_2 & \cdots & f_M \end{bmatrix}^T \\ \boldsymbol{g} &= \begin{bmatrix} g_1 & g_2 & \cdots & g_M \end{bmatrix}^T.\end{aligned} \tag{10}$$

Here, $\boldsymbol{e}$ is the amplitude-error vector, $\boldsymbol{f}$ is the desired-amplitude vector, $\boldsymbol{g}$ denotes the designed-amplitude vector, and $\boldsymbol{W}$ is a diagonal matrix containing weighting coefficients. The components of vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ are defined by

$$f_m = A_d(\omega_m, \rho_n)$$
$$g_m = |H(\omega_m)|$$

and ω_m denotes the mth sample of frequency ω, $H(\omega_m)$ denotes the mth sample of actual frequency-response $H(\omega)$. For each n, an OO-CBW filter is designed, which results in a set of coefficients $\{a_i, x_{i2}, x_{i1}\}$. By repeating the above design procedures for $n = 1, 2, \cdots, N$, we finally yield N sets of coefficients $\{a_i, x_{i2}, x_{i1}\}$. The above procedures belong to the first-stage design. Then, the second-stage is to fit an individual polynomial in ρ to the resulting values of each coefficient of $\{a_i, x_{i2}, x_{i1}\}$. Performing the least-squares fit yields all the approximating polynomials. Consequently, this second-step gives the approximating polynomials $\{a_i(\rho), x_{i2}(\rho), x_{i1}(\rho)\}$.

IV. Odd-Order Example

Let us approximate the desired bandpass amplitude

$$A_d(\omega,\rho) = \begin{cases}0, & |\omega| \in [0, \rho + 0.26\pi] \\ 1, & |\omega| \in [\rho + 0.32\pi, 0.64\pi - \rho] \\ 0, & |\omega| \in [0.70\pi - \rho, \pi]\end{cases} \tag{11}$$

with

$$\rho \in [\rho_{\min}, \rho_{\max}] = [-0.10\pi, 0.10\pi].$$

$A_d(\omega,\rho)$ is best fitted by setting the design parameters

$$\begin{aligned}&(N_1, N_2) = (6, 3) \\ &\gamma = 0.9999 \\ &C(x) = \cos(x) \\ &(M, N) = (601, 21) \\ &p = 10.\end{aligned} \tag{12}$$

The weighting matrix $\boldsymbol{W}$ is set as a diagonal matrix

$$\boldsymbol{W} = \mathrm{diag}(w_1, w_1, \cdots, w_M) \tag{13}$$

whose components are

$$w_m = \begin{cases}1, & \omega_m \in \text{passband and stopband} \\ 0, & \omega_m \in \text{transition band.}\end{cases} \tag{14}$$

By using the above weights, the design errors in the transition bands can be ignored. Therefore, the approximations of the passband and stopband responses can be emphasized.

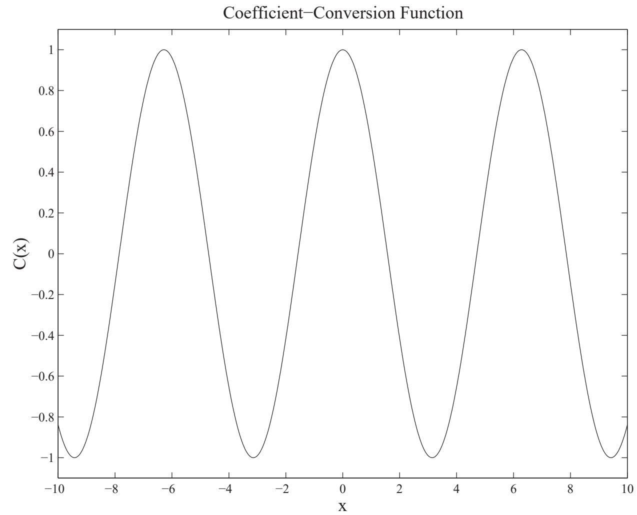

Fig. 1. $C(x)$ used for conversions.

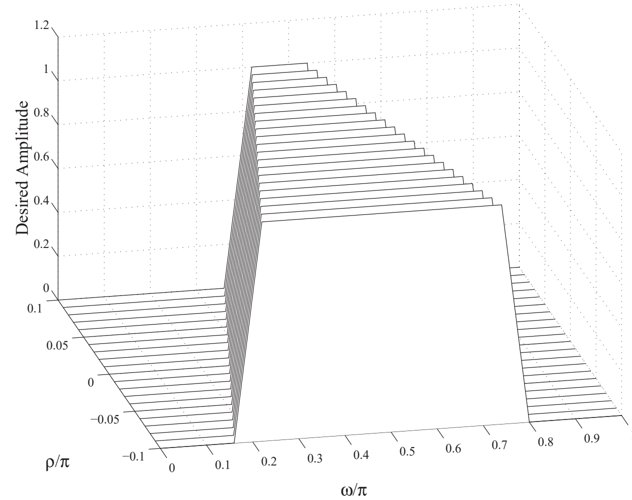

Fig. 2. Discretized bandpass specifications ($N = 21$).

Fig. 1 shows the CC function $C(x)$, and Fig. 2 plots the discretized bandpass specifications.

The OO-CBW filter corresponding to $\rho_1 = \rho_{\min} = -0.10\pi$ is first designed. The initial coefficient values used for designing this first OO-CBW filter are

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ x_{12} \\ x_{22} \\ x_{32} \\ x_{01} \\ x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} -0.13639588308660 \\ 0.11393131352081 \\ 1.06676821135919 \\ 0.05928146052361 \\ -0.09564840548367 \\ -0.83234946365002 \\ 0.29441081639264 \\ -1.33618185793780 \\ 0.71432455181895 \\ 1.62356206444627 \\ -0.69177570170229 \\ 0.85799667282826 \\ 1.25400142160253 \\ -1.59372957644748 \end{bmatrix}. \tag{15}$$

After the first OO-CBW filter is designed, its results (coefficient values) are utilized for starting designing the next one. In total, 21 OO-CBW filters need to be designed. It should be mentioned that designing an OO-CBW filter needs to solve a nonlinear optimization problem, which can be done by employing *fminsearch* in MATLAB.

Fig. 3 plots the OO-CBW filter's amplitude responses for $N = 21$. The average normalized rms error and the average maximum error are

$$\begin{aligned} \bar{e}_2 &= 5.4733\% \\ \bar{e}_{\max} &= 0.0618 \end{aligned} \tag{16}$$

respectively.

After designing the OO-CBW filters ($N = 21$), we fit an individual third-order polynomial to the values of each coefficient. Fig. 4 shows the plots of the approximating polynomials $a_i(\rho)$, and Fig. 5 shows the plots of the approximating polynomials $x_{i2}(\rho), x_{i1}(\rho)$, where $x_{02}(\rho) = 0$. Moreover, Fig. 6 shows the amplitude responses of the OO-VBW filter, Fig. 7 plots the amplitude-response errors with transition-band erros being ignored. The average normalized rms error and average maximum error are respectively

$$\begin{aligned} \bar{e}_2 &= 5.5199\% \\ \bar{e}_{\max} &= 0.0641. \end{aligned} \tag{17}$$

Finally, the filter stability must be checked. Fig. 8 shows the stability triangles along with the loci of the denominator-coefficient pairs (b_{i1}, b_{i2}) when the value of ρ is changed. The figures make it clear that all (b_{i1}, b_{i2}) are always moving within the inner areas of the triangles when ρ is varied. Therefore, the designed OO-VBW filter is absolutely stable.

V. Conclusion

This paper has extended the even-order L_p-norm design to the OO-VBW filter case, and the resulting OO-VBW filter keeps always stable when as the BW-varying parameter varies. A bandpass example has demonstrated the filter accuracy and the ensured stability.

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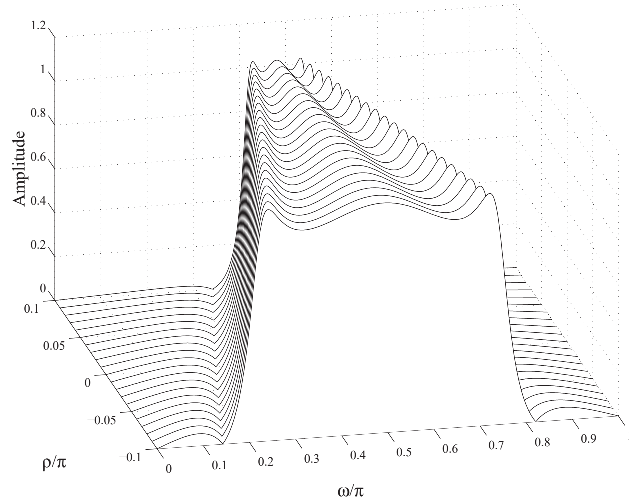

Fig. 3. Bandpass responses of the CBW filters ($N = 21$).

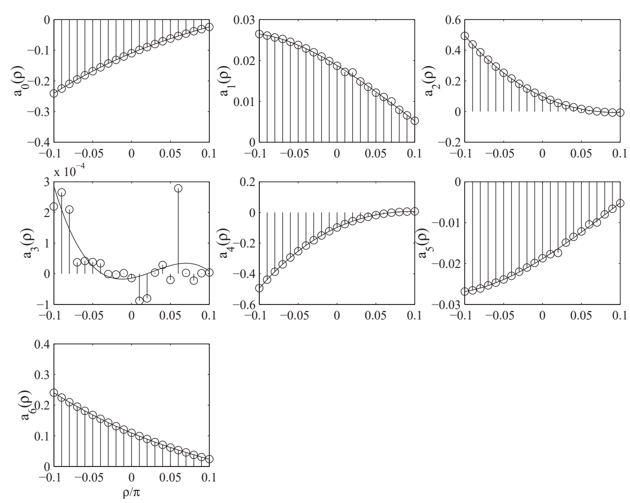

Fig. 4. Third-order polynomials $a_i(\rho)$.

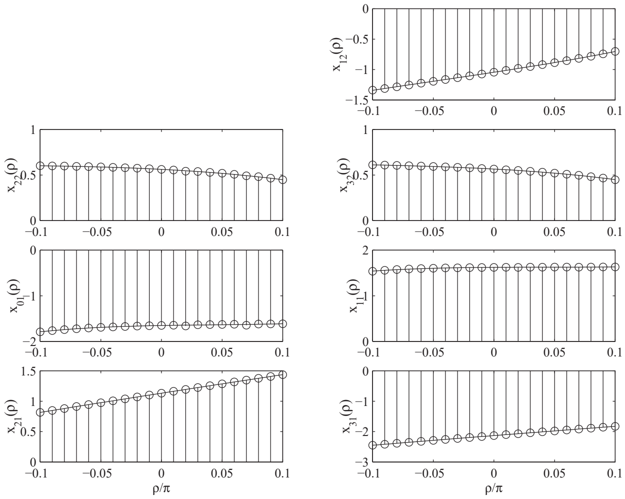

Fig. 5. Third-order polynomials $x_{i2}(\rho)$ and $x_{i1}(\rho)$.

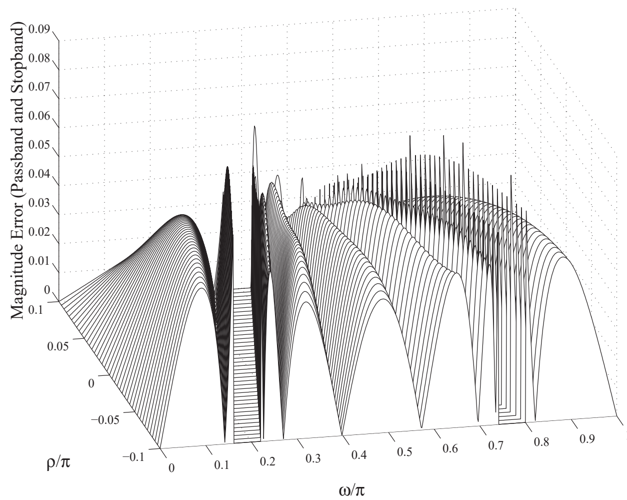

Fig. 7. Amplitude errors (passband and stopband only).

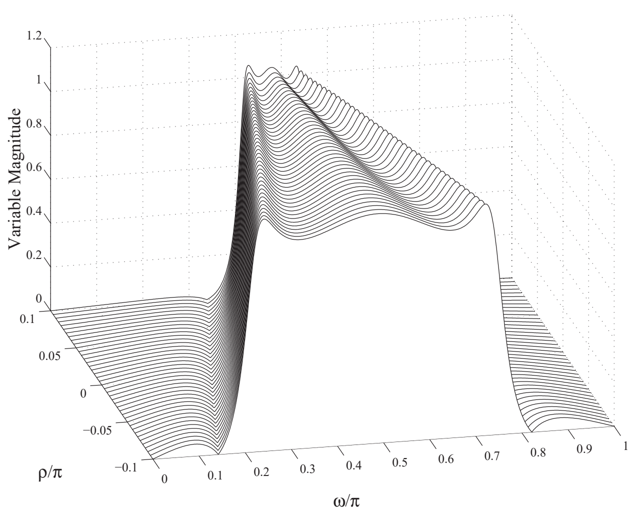

Fig. 6. Amplitude responses (OO-VBW filter).

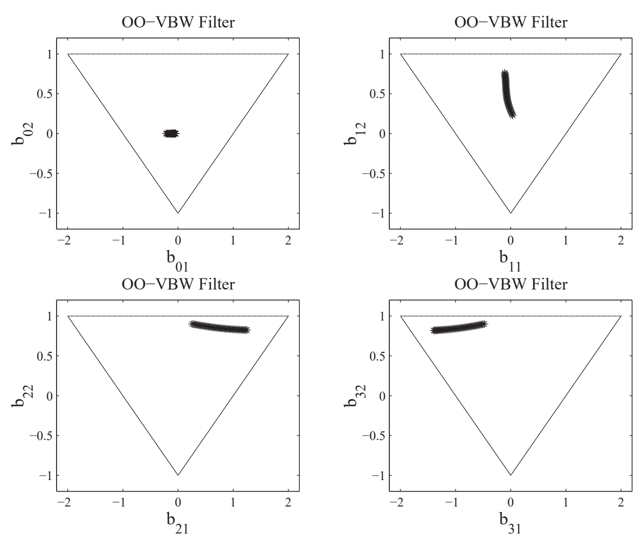

Fig. 8. Stability triangles and the loci of pairs (b_{i1}, b_{i2}).